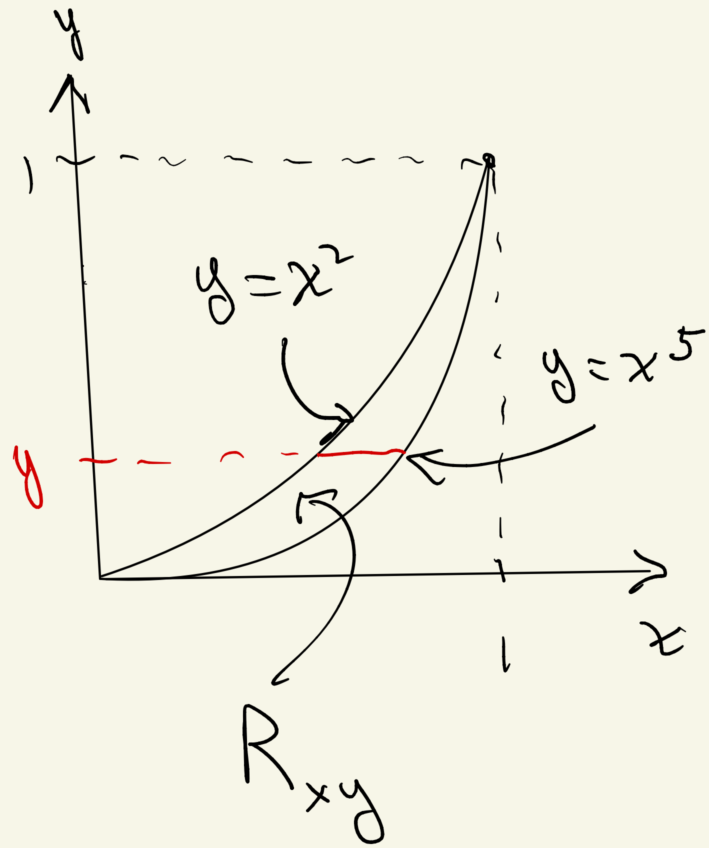
Math 210 Midterm I Solutions
Winter 2023

#1 (a) $\displaystyle\int_0^1 \int_{x^5}^{x^2} dy\,dx$



(b) $\displaystyle\int_0^1 \int_{\sqrt{y}}^{\sqrt[5]{y}} dx\,dy$

(2)

#2 (a) $\bar{x} = \dfrac{M_y}{M}$ $\quad \bar{y} = \dfrac{M_x}{M}$

$$M = \iint_{R_{xy}} \delta \, dA = \int_0^1 \int_{x^5}^{x^2} x^2 y^3 \, dy \, dx$$

$$M_y = \iint_{R_{xy}} x \, dA = \int_0^1 \int_{x^5}^{x^2} x^3 y^3 \, dy \, dx$$

$$M_x = \iint_{R_{xy}} y \, dA = \int_0^1 \int_{x^5}^{x^2} x^2 y^4 \, dy \, dx$$

(b) $KE = \frac{1}{2} I_{y=1} \omega^2$

$$I_{y=1} = \iint_{R_{xy}} (x-1)^2 \delta \, dA = \int_0^1 \int_{x^5}^{x^2} (x-1)^2 x^2 y^3 \, dy \, dx$$

#3 a $x = r\cos\theta \quad y = r\sin\theta$

$$J = \det\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r \quad \checkmark$$

b Since $\det\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| > 0$, then

(r,θ) is positively oriented wrt (x,y)

But $\det\left|\frac{\partial(x,y)}{\partial(\theta,r)}\right| = \begin{vmatrix} x_\theta & x_r \\ x_\theta & x_r \end{vmatrix}$

$$= \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} = -r$$

so (θ, r) is negatively oriented wrt (x,y)

#4

$$f(x,y,z) = \frac{1}{r^2} = r^{-2}$$

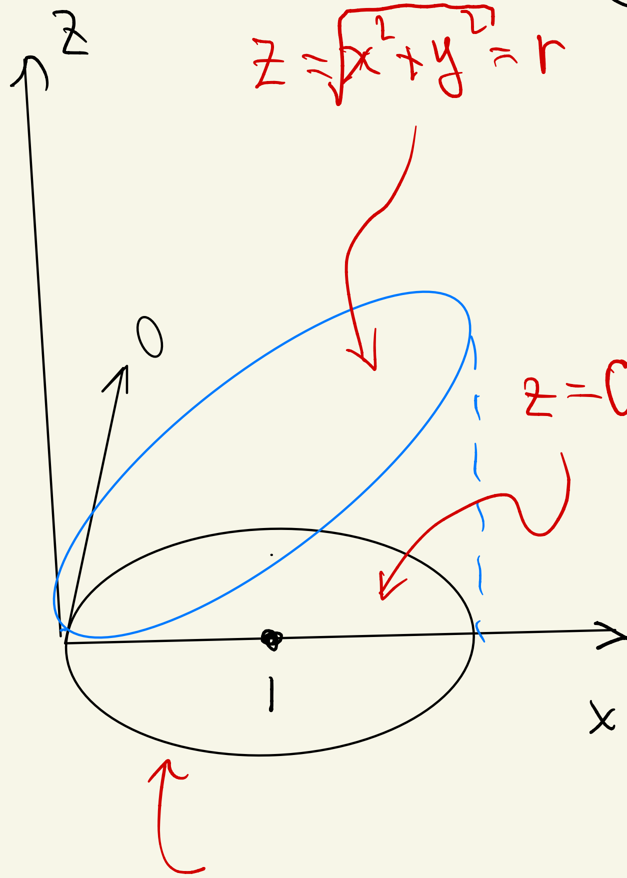
$$I = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_0^{r} r^{-2}\, r\, dz\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^{-1} z \Big]_{z=0}^{z=r} dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^{-1}\cdot r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 2\cos\theta\, d\theta$$

$$= 2\sin\theta\Big]_{-\pi/2}^{\pi/2} = 2(1-(-1)) = 2\cdot 2 = 4$$



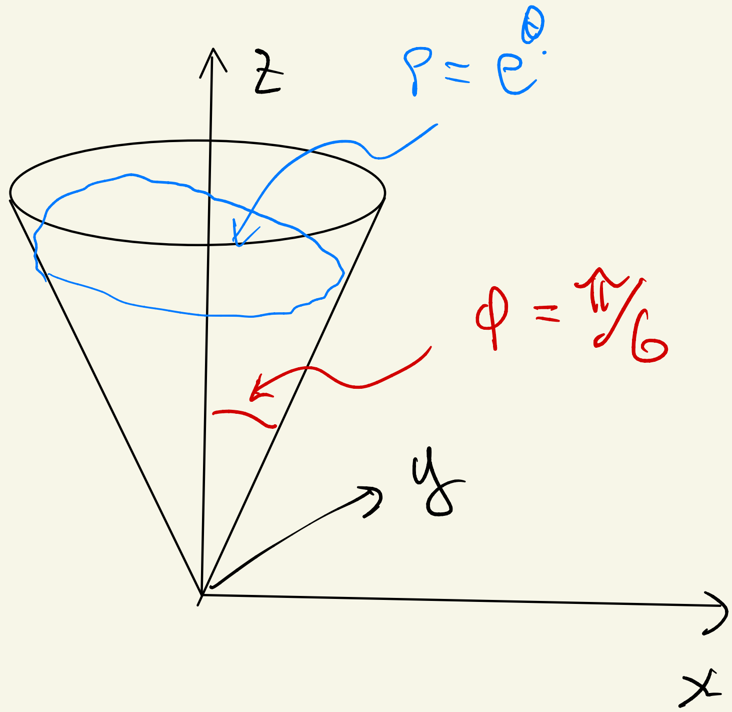
$$(x-1)^2 + y^2 = 1$$

$$x^2 - 2x + \cancel{1} + y^2 = \cancel{1}$$

$$r^2 = 2x = 2r\cos\theta$$

$$\boxed{r = 2\cos\theta}$$

#5 (a)



$$I = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{e^\theta} \delta \rho^2 \sin^2\varphi \, d\rho\, d\varphi\, d\theta$$

b $\delta = \dfrac{1}{\sin^2\varphi}$ so Mass M in kg is

$$M = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{e^\theta} \rho^2 \, d\rho \, d\varphi \, d\theta$$

$$\left.\frac{\rho^3}{3}\right]_0^{e^\theta} = \frac{e^{3\theta}-1}{3}$$

$$= \int_0^{2\pi}\int_0^{\pi/6} \frac{e^{3\theta}-1}{3} \, d\varphi \, d\theta = \frac{\pi}{6}\int_0^{2\pi} \frac{e^{3\theta}-1}{3} \, d\theta$$

$$= \frac{\pi}{6}\left[\frac{\frac{1}{3}e^{3\theta}-\theta}{3}\right]_0^{2\pi} = \boxed{\frac{\pi}{6}\left(\frac{1}{9}e^{6\pi} - \frac{2\pi+1}{3}\right)}$$

in kg